Scalar Product

Fact — The **scalar product** of two vectors $\mathbf{a}$, $\mathbf{b}$ is denoted by $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ as well as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} p \\ q \\ r \end{pmatrix} = ap + bq + cr$$

Fact — Two vectors are perpendicular if $\mathbf{a} \cdot \mathbf{b} = 0$

Example

Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$

Equation of a line

Example

Find the vector equation of the line $x = 2y = 7z$

Suppose $x = \frac{y}{2} = \frac{z}{7} = \lambda$, *then*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \\ 7\lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

Example

Find the vector equation of the line $\frac{x-1}{3} = \frac{y+1}{4} = \frac{z-3}{5}$

Suppose $\frac{x-1}{3} = \frac{y+1}{4} = \frac{z-3}{5} = \lambda$, *then*

$$\begin{pmatrix} x-1 \\ y+1 \\ z-3 \end{pmatrix} = \begin{pmatrix} 3\lambda \\ 4\lambda \\ 5\lambda \end{pmatrix} = \lambda \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

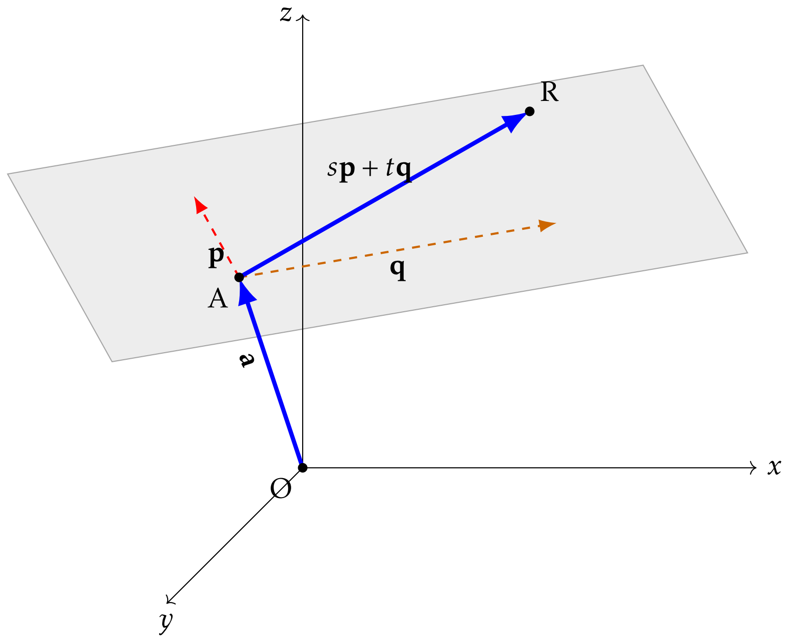
Example

Find the cartesian equation of the line $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\frac{x-5}{1} = \frac{y-2}{-1} = \frac{z+1}{1}$

What happens if one the variables in the direction of the line is 0?

Equation of a plane



Fact — We can express every point on the plane through point $\mathbf{a}$, with non-zero, non-parallel vectors $\mathbf{p}, \mathbf{q}$ in the form $\mathbf{r} = \mathbf{a} + s\mathbf{p} + t\mathbf{q}$

Example

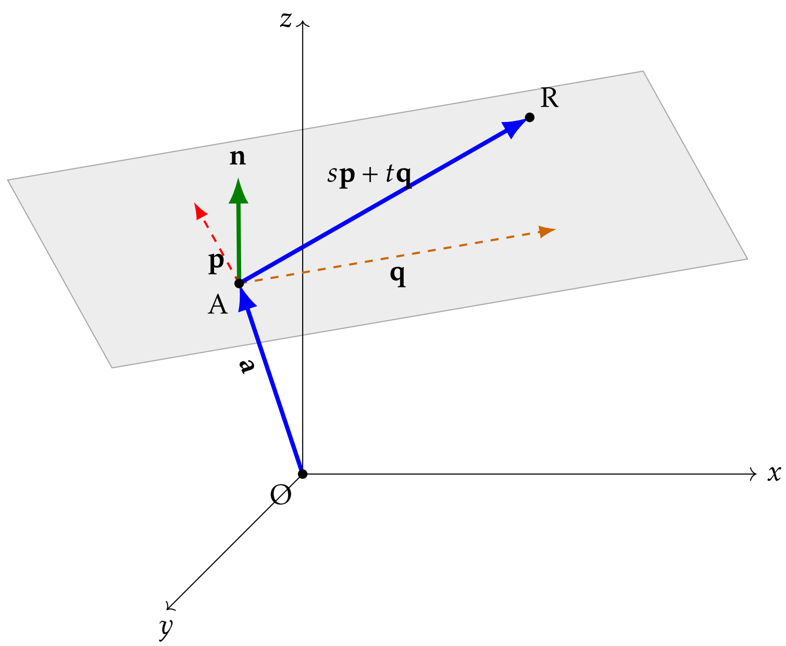
Find a vector equation of the plane through the point $A(1,1,1)$, $B(1,-3,2)$ and $C(1,0,1)$

Example

Find a cartesian equation of the plane

$$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}1\\-1\\-1\end{pmatrix} + t\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$$\begin{aligned}
& \begin{pmatrix}x\\y\\x\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}1\\-1\\-1\end{pmatrix} + t\begin{pmatrix}2\\-1\\1\end{pmatrix} \\
\Rightarrow \quad & x = 1 + s + 2t \\
& y = 2 - s - t \\
& z = 3 - s + t \\
\Rightarrow \quad & x + y = 3 + t \\
\Rightarrow \quad & z - y = 1 + 2t \\
\Rightarrow \quad & 2(x+y) - (z-y) = 5 \\
& 2x + 3y - z = 5
\end{aligned}$$



Notice that $\mathbf{n}$ *is perpendicular to all the vectors* $\mathbf{r}-\mathbf{a}$ *in the plane, so in particular* $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$*. If we write* $\mathbf{r} = \langle x, y, z\rangle, \mathbf{n} = \langle p, q, r\rangle$ *then the equation becomes* $px + qy + rz = \mathbf{a}\cdot\mathbf{n}$ *the form we have already found!*

Fact — Three equations for a plane:

1. *Vector equation*: $\mathbf{r} = \mathbf{a} + s\mathbf{p} + t\mathbf{q}$
2. *Normal equation*: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
3. *Cartesian equation*: $px + qy + rz = k$

Example

Find the cartesian equation of the plane through the point $(1,2,3)$ with normal $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

Method 1: *The equation is* $4x + 5y + 6z = k$. *We have that* $(1,2,3)$ *is on the line, so* $k = 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 = 32$ *ie* $4x + 5y + 6z = 32$.

Method 2:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$
$$4x + 5y + 6z = 32$$

Angles

Angle between two lines

Recall that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between the two vectors. So we can find the angle between two lines:

Example
Find the angle between the line joining $(1,2)$ and $(3,-5)$ and the line joining $(2,-3)$ to $(1,4)$.

The vectors are $\begin{pmatrix} 2 \\ -7 \end{pmatrix}, \begin{pmatrix} 1 \\ -7 \end{pmatrix}$,

$$\begin{aligned} & \sqrt{2^2 + (-7)^2}\sqrt{1^2 + (-7)^2}\cos\theta = (2 \cdot 1) + ((-7) \cdot (-7)) \\ \Rightarrow \quad & \sqrt{53}\sqrt{50}\cos\theta = 51 \\ \Rightarrow \quad & \cos\theta = \frac{51}{\sqrt{53}\sqrt{50}} \\ \Rightarrow \quad & \theta = 7.8^\circ \end{aligned}$$

Example
Find the angle between the line joining $(1,3,-2)$ and $(2,5,-1)$ and the line joining $(-1,4,3)$ to $(3,2,1)$:

The vectors are $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$ *and so*

$$\begin{aligned} & \sqrt{1^2 + 2^2 + 1^2}\sqrt{4^2 + (-2)^2 + (-2)^2}\cos\theta = 1 \cdot 4 + 2 \cdot (-2) + 1 \cdot (-2) \\ \Rightarrow \quad & \sqrt{6}\sqrt{24}\cos\theta = -2 \\ \Rightarrow \quad & 12\cos\theta = -2 \\ \Rightarrow \quad & \cos\theta = -\frac{1}{6} \\ \Rightarrow \quad & \theta = 99.6^\circ \end{aligned}$$

So the angle between them is 80.4°

Example

Find the angle between the diagonals of a cube.

One vector (putting one corner in at the origin) is $\langle 1,1,1\rangle$*, the other vector will be* $\langle 1,-1,-1\rangle$*, and so*

$$\begin{aligned} \sqrt{1^2+1^2+1^2}\sqrt{1^2+(-1)^2+(-1)^2}\cos\theta &= 1\cdot 1+1\cdot(-1)+1\cdot(-1) \\ \Rightarrow \qquad \cos\theta &= -\frac{1}{3} \\ \Rightarrow \qquad \theta &= 109.5^\circ \end{aligned}$$

Therefore the angle between them is 70.5°.

Angle between a line and a plane

Example

Find the acute angle between the line $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-5}{1}$ and the plane $x-y+z=0$.

Notice that if we find the angle between the plane and the line, then

Sketch $\mathbf{n}$ *(the normal to the plane) and* $\mathbf{p}$ *(the direction of the line) and notice that we are looking for* $\frac{\pi}{2}-\theta$*. We can then compute:*

$\mathbf{n}=\begin{pmatrix}1\\-1\\1\end{pmatrix}, \mathbf{p}=\begin{pmatrix}2\\4\\1\end{pmatrix}$

and so $\sqrt{3}\sqrt{21}\cos\theta=-1 \Rightarrow \cos\theta=-\frac{1}{3\sqrt{7}}$

Angle between two planes

Example

A pyramid of height 3 units stands symmetrically on a rectangular base $ABCD$ with $AB = 2$ units and $BC = 4$ units. Find the angle between slanting faces

Vector product

Fact — The **vector product** of two vectors $\mathbf{a}$, $\mathbf{b}$ is given by:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{ab}$ in the right hand sense.

Example

Find the vector products

(a) $\mathbf{i} \times \mathbf{j}$

(b) $\mathbf{k} \times \mathbf{k}$

(c) $\mathbf{j} \times \mathbf{i}$

(a) The magnitude of $\mathbf{i} \times \mathbf{j}$ *is* $1 \times 1 \times \sin\frac{\pi}{2} = 1$ *and it's direction is* $\mathbf{k}$. *A vector of magnitude* 1 *in the* $\mathbf{k}$*-direction is* $\mathbf{k}$. *Therefore* $\mathbf{i} \times \mathbf{j} = \mathbf{k}$

(b) The magnitude of $\mathbf{k} \times \mathbf{k}$ *is* $1 \times 1 \times \sin 0 = 0$, *so* $\mathbf{k} \times \mathbf{k} = 0$

(c) The magnitude of $\mathbf{j} \times \mathbf{i}$ *is* $1 \times 1 \times \sin\frac{\pi}{2} = 1$ *and it's direction is* $-\mathbf{k}$. *A vector of magnitude* 1 *in the* $-\mathbf{k}$*-direction is* $-\mathbf{k}$. *Therefore* $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

Fact — For vectors, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ and scalar s:

$$\mathbf{p} \times \mathbf{q} = -\mathbf{q} \times \mathbf{p}$$
$$s(\mathbf{p} \times \mathbf{q}) = (s\mathbf{p}) \times \mathbf{q} = \mathbf{p} \times (s\mathbf{q})$$
$$(\mathbf{p} + \mathbf{q}) \times \mathbf{r} = \mathbf{p} \times \mathbf{r} + \mathbf{q} \times \mathbf{r}$$

Example

Compute $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\begin{aligned}
\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + \\
&\quad a_2\mathbf{j} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + \\
&\qquad a_3\mathbf{k} \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1b_1(\mathbf{i} \times \mathbf{i}) + a_1b_2(\mathbf{i} \times \mathbf{j}) + a_1b_3(\mathbf{i} \times \mathbf{k}) + \\
&\quad a_2b_1(\mathbf{j} \times \mathbf{i}) + a_2b_2(\mathbf{j} \times \mathbf{j}) + a_2b_3(\mathbf{j} \times \mathbf{k}) + \\
&\qquad a_3b_1(\mathbf{k} \times \mathbf{i}) + a_3b_2(\mathbf{k} \times \mathbf{j}) + a_3b_3(\mathbf{k} \times \mathbf{k}) \\
&= a_1b_1\mathbf{0} + a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + \\
&\quad a_2b_1(-\mathbf{k}) + a_2b_2\mathbf{0} + a_2b_3\mathbf{i} + \\
&\qquad a_3b_1\mathbf{j} + a_3b_2(-\mathbf{i}) + a_3b_3\mathbf{0} \\
&= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\
&= \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\
&= \begin{vmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}
\end{aligned}$$

Example

Find the vector product $\begin{pmatrix}1\\-3\\-5\end{pmatrix}\times\begin{pmatrix}0\\-1\\-2\end{pmatrix}$

$$\begin{pmatrix}1\\-3\\-5\end{pmatrix}\times\begin{pmatrix}0\\-1\\-2\end{pmatrix}=\begin{pmatrix}(-3)\cdot(-2)-(-5)\cdot(-1)\\(-5)\cdot 0-1\cdot(-2)\\1\cdot(-1)-(-3)\cdot 0\end{pmatrix}$$
$$=\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

Notice that $\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\-5\end{pmatrix}=1-6+5=0$ *and* $\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\-2\end{pmatrix}=0-2+2=0$

Example

Find the area of the triangle ABC with vertices $A(1,2,3)$, $B(4,6,2)$ and $C(6,8,10)$.

Since the area of a triangle is $\frac{1}{2}AB\times AC\times\sin\theta$ *the area is* $\frac{1}{2}|\vec{AB}\times\vec{AC}|$, *ie*

$$\vec{AB}=\begin{pmatrix}4\\6\\2\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}3\\4\\-1\end{pmatrix}$$
$$\vec{AC}=\begin{pmatrix}6\\8\\10\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}5\\6\\7\end{pmatrix}$$
$$\vec{AB}\times\vec{AC}=\begin{pmatrix}3\\4\\-1\end{pmatrix}\times\begin{pmatrix}5\\6\\7\end{pmatrix}$$
$$=\begin{pmatrix}34\\-26\\-2\end{pmatrix}$$

Therefore the area is $\frac{1}{2}\sqrt{34^2+(-26)^2+(-2)^2}=3\sqrt{51}$

Example

Find the normal equation of the plane

$$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}1\\-1\\-1\end{pmatrix} + t\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

$$\begin{aligned}\hat{\mathbf{n}} &= \begin{pmatrix}1\\-1\\-1\end{pmatrix} \times \begin{pmatrix}2\\-1\\1\end{pmatrix}\\ &= \begin{pmatrix}-1-1\\-2-1\\-1+2\end{pmatrix} = \begin{pmatrix}-2\\-3\\1\end{pmatrix}\end{aligned}$$

Therefore we have the equation $\begin{pmatrix}-2\\-3\\1\end{pmatrix} \cdot \mathbf{r} = \begin{pmatrix}-2\\-3\\1\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} = -5$

Example

Find a vector equation for the line

$$\mathbf{r} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

For any point on the line $\mathbf{r} - \begin{pmatrix}1\\2\\4\end{pmatrix}$ *is parallel to* $\begin{pmatrix}1\\-1\\1\end{pmatrix}$*, in particular*

$$\left(\mathbf{r} - \begin{pmatrix}1\\2\\4\end{pmatrix}\right) \times \begin{pmatrix}1\\-1\\1\end{pmatrix} = 0$$

Fact — The vector equation of a line with direction vector $\mathbf{d}$ and point on the line $\mathbf{a}$ can be written as $(\mathbf{r} - \mathbf{a}) \times \mathbf{d} = 0$

Scalar Triple Product

Example
What is the area of a triangle?

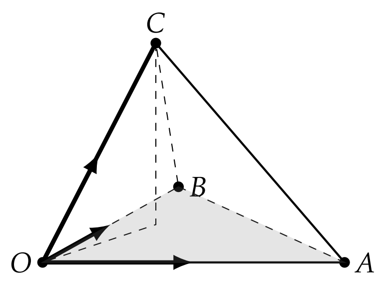
$\frac{1}{2}ab\sin C = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

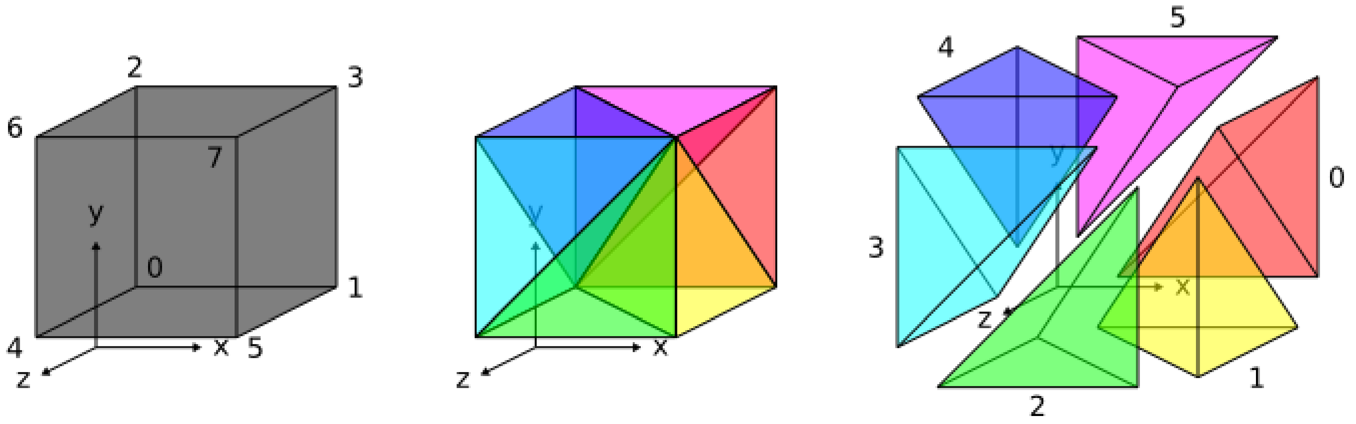
Example
What is the area of a parallelogram?

$2\cdot\frac{1}{2}|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}\times\mathbf{b}|$

Example
What is the volume of a tetrahedron?

The triangle is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$. *The height is* $h = |c|\cos\theta = \mathbf{c}\cdot\mathbf{n}$. *Therefore the volume is* $\frac{1}{3}\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$





> **Example**
> What is the volume of a parallelepiped?

The volume is $6 \cdot \frac{1}{6}\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$

Definition. The **scalar triple product** of 3 vectors is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

> Fact —
> $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

> **Example**
> Calculate the volume of a tetrahedron with vertices $P(1,3,2), Q(4,4,2), R(2,6,2)$ and $S = (3,5,7)$.

Fact — If we have $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Example

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & a \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$. Find the value of a for which it is singular

$$\begin{aligned} \det(\mathbf{A}) &= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} a \\ 1 \\ 4 \end{pmatrix} \right) \\ &= \begin{pmatrix} a \\ 1 \\ 4 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} a \\ 1 \\ 4 \end{pmatrix} \cdot \left(\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right) \\ &= -3 - a \\ \Rightarrow \qquad a &= -3 \end{aligned}$$

Intersections

Intersection of two lines

If we have two lines in two dimensions then

- They are parallel
  - They are the same line
  - They never meet
- They meet at a single point

If we have two lines in three (or more) dimensions then:

- They are parallel
  - They are the same line
  - They never meet
- They meed at a single point
- They are skew

Example

Find the cartesian equations of the line l joining $(-1,4,1)$ to $(3,6,2)$, and find whether this line intersects the line m with cartesian equation $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{-2}$

The direction vector of the line l is $\begin{pmatrix}3\\6\\2\end{pmatrix} - \begin{pmatrix}-1\\4\\1\end{pmatrix} = \begin{pmatrix}4\\2\\1\end{pmatrix}$ *therefore the equation of the line is* $\frac{x+1}{4} = \frac{y-4}{2} = \frac{z-1}{1}$.
To find the intersection of the line, use vector form:
$\mathbf{r} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + t\begin{pmatrix}4\\2\\1\end{pmatrix}$ *and* $\mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + t\begin{pmatrix}1\\2\\-2\end{pmatrix}$. *Suppose they meet at a point with parameters r, s then*

$$\begin{aligned}
& \begin{pmatrix}-1\\4\\1\end{pmatrix} + r\begin{pmatrix}4\\2\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + s\begin{pmatrix}1\\2\\-2\end{pmatrix} \\
\Rightarrow \quad & -1 + 4r = 1 + s \\
& 4 + 2r = 1 + 2s \\
& 1 + t = 1 - 2s \\
\Rightarrow \quad & 4r - s = 2 \\
& 2r - 2s = -3 \\
& t + 2s = 0
\end{aligned}$$

But this system is inconsistent, therefore the lines do not meet.

Intersection of three planes

> **Fact —** $x + 2y + 3z = 6$ defines a plane, or in general $ax + by + cz = k$ defines a plane.

Therefore if we are looking for the intersection of three planes, we are looking for solutions to a system of simultaneous equations in 3 unknowns.

We have already seen that there are many different things which can happen when solving simultaneous equations in 3 variables:

- Unique solution
- No solutions
  - The 3 planes form a prism
  - The 3 planes are all parallel and not the same
- A line of solutions
  - The 3 planes form a sheaf (all distint planes around a point)
  - Two of the planes are the same and the other plane intersects it

Intersection of two planes

Similar to the intersection of three planes, except we cannot have a unique solution. They are either parallel or meet at a line.

Intersection of a line and plane

Example

Find the intersection between the given line and plane, or show they do not intersect

(a) $\mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$ and $2x - y + 2z = 5$

(b) $\frac{x-1}{-1} = \frac{y}{-3} = \frac{z+4}{2}$ and $x - 3y - 4z = 12$

1.

$$\begin{aligned} 5 &= 2 \cdot (4 + 3\lambda) - (-3) + 2(1 - 2\lambda) \\ &= 8 + 6\lambda + 3 + 2 - 4\lambda \\ &= 2\lambda + 13 \\ \lambda &= -4 \end{aligned}$$

Therefore a unique intersection point at $(-8, -3, 9)$

2.

$$\begin{aligned} x &= 1 + -\mu \\ y &= -3\mu \\ z &= 2\mu - 4 \\ \Rightarrow \quad (1 - \mu) - 3 \cdot (-3\mu) - 4 \cdot (2\mu - 4) &= 12 \\ 17 &= 12 \end{aligned}$$

Therefore no solutions.

Example

Show that the line $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ lies in the plane $x - 3y = 6$

$$(3 + 3t) - 3(-1 + t) = 6$$

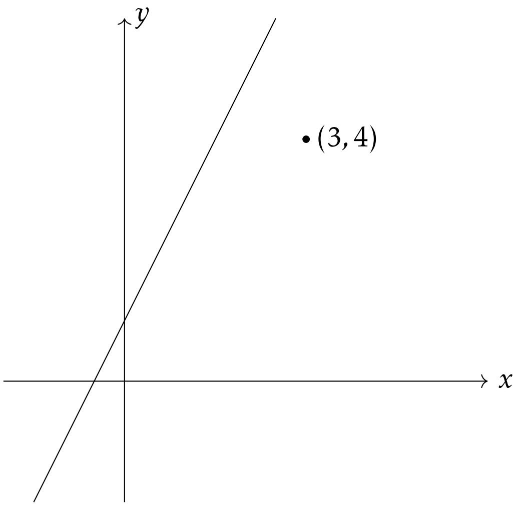
Therefore every t is a solution so every point on the line lines on the plane.

Shortest distances

Shortest distance between a point and a line in 2D

Example

What is the shortest distance between the line $y = 2x + 1$ and the point $(3, 4)$?



Notice that the shortest distance will be when the line perpendicular to $y = 2x + 1$ goes through $(3, 4)$. We know this line will have the form $2y + x = c$, ie $2y + x = 11$.

Solving the simultaneous equations, we can find the point is $\left(\frac{9}{5}, \frac{23}{5}\right)$ and the distance will be $\sqrt{\left(\frac{9}{5} - 3\right)^2 + \left(\frac{23}{5} - 4\right)^2} = \frac{3\sqrt{5}}{5}$

Example

What is the shortest distance between the line $ax + by = c$ and the point (x_1, y_1)?

Notice the perpendicular line is $bx - ay = d$ or $bx - ay = bx_1 - ay_1$. Solving the simultaneous equations:

$$
\begin{aligned}
ax + by &= c \\
bx - ay &= bx_1 - ay_1
\end{aligned}
$$

we obtain $(x, y) = \frac{ac+b^2x_1-aby_1}{a^2+b^2}, \frac{bc-abx_1+a^2y}{a^2+b^2}$ and therefore the distance is:

$$
\begin{aligned}
D &= \sqrt{\left(\frac{ac + b^2x_1 - aby_1}{a^2 + b^2} - x_1\right)^2 + \left(\frac{bc - abx_1 + a^2y}{a^2 + b^2} - y_1\right)^2} \\
&= \sqrt{\left(\frac{ac - a^2x_1 - aby_1}{a^2 + b^2}\right)^2 + \left(\frac{bc - abx_1 - b^2y_1}{a^2 + b^2}\right)^2} \\
&= \sqrt{\frac{a^2}{(a^2 + b^2)^2}(c - ax_1 - by_1)^2 + \frac{b^2}{(a^2 + b^2)^2}(c - ax_1 - by_1)^2} \\
&= \frac{|c - ax_1 - by_1|}{\sqrt{a^2 + b^2}}
\end{aligned}
$$

Shortest distance between a point and a line in 3D

Example

Find the shortest distance between the line $\mathbf{r} = \begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and the point $P(1,0,-3)$

Method 1: The point P must lie in the plane perpendicular to $\begin{pmatrix}-1\\1\\2\end{pmatrix}$. *This plane must be* $\begin{pmatrix}-1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-3\end{pmatrix} = -7$.

The plane and the line intersect at $(-1)\cdot(1-\lambda)+1\cdot(2+\lambda)+2\cdot(2\lambda) = -7 \Rightarrow \lambda = -\frac{4}{3}$. *Therefore the point is* $(\frac{7}{3}, \frac{2}{3}, -\frac{8}{3})$ *and the distance between this and P is* $\frac{\sqrt{21}}{3}$

Method 2: The distance D, between $\begin{pmatrix}1-\lambda\\2+\lambda\\2\lambda\end{pmatrix}$ *and* $\begin{pmatrix}1\\0\\-3\end{pmatrix}$ *satisfies* $D^2 = (1-\lambda-1)^2+(2+\lambda)^2+(2\lambda+3)^2 = 6\lambda^2+16\lambda+13$ *which is minimized at* $\lambda = -\frac{4}{3}$ *and* $D = \frac{\sqrt{21}}{3}$

Shortest distance between a point and a plane

Example

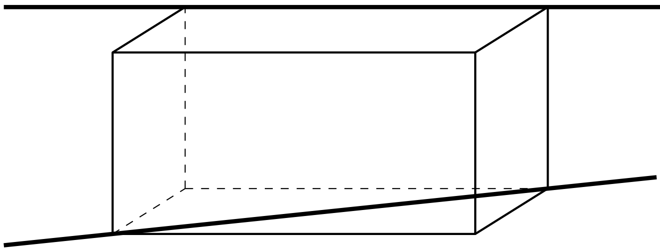
Find the perpendicular distance of the point $P(4,5,6)$ from the plane $x+2y-2z=9$.

The plane has normal $\mathbf{n}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$. *The length of the perpendicular to the plane is the projection of the vector between the point P and point on the plane onto the normal, ie*

$\mathbf{n}\cdot(\mathbf{p}-\mathbf{a})$

Shortest distance between two lines

Example
Find the shortest distance between the lines $\mathbf{r} = \mathbf{a}_1 + s\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + s\mathbf{b}_2$



Notice that the shortest distance is going to be in the corner PQ. Notice also the vector $\vec{PQ}$ is perpendicular to both $\mathbf{b}_1$ and $\mathbf{b}_2$, ie it's parallel to $\mathbf{b}_1 \times \mathbf{b}_2$
The length AN is

$$\begin{aligned} A_1A_2\cos\theta &= |\mathbf{a}_1 - \mathbf{a}_2|\cos\theta \\ &= |(\mathbf{a}_1 - \mathbf{a}_2)\cdot\mathbf{n}|/|\mathbf{n}| \\ &= \frac{|(\mathbf{a}_1 - \mathbf{a}_2)\cdot(\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|} \end{aligned}$$

Fact — The shortest distance between two lines, given by $\mathbf{r} = \mathbf{a}_1 + s\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + s\mathbf{b}_2$ is

$$d = \frac{|(\mathbf{a}_1 - \mathbf{a}_2)\cdot(\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$